Resit Exam — Partial Differential Equations (WBMA008-05)

Wednesday 9 July 2025, 15.00–17.00h

University of Groningen

Instructions

1. The use of calculators is *not* allowed. It is allowed to use a "cheat sheet" with notes (one sheet A4, both sides, handwritten,"wet ink").
2. All answers need to be accompanied with an explanation or a calculation: only answering "yes", "no", or "42" is not sufficient.
3. If p is the number of marks then the grade is $G = 1 + p/10$.

Problem 1 (8 + 6 + 6 = 20 points)

Consider the following nonlinear transport equation:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0, \quad u(0,x) = f(x).$$

(a) Show that the solution u is constant along the characteristic curves satisfying the equation

$$\frac{dx}{dt} = u(t, x(t)).$$

(b) Determine the characteristic curve through $(0, y)$; express the answer in terms of f and y.

(c) Assume $f(x) = x$. Explicitly compute the solution $u(t,x)$ for all $t \geq 0$.

Problem 2 (20 points)

Consider the following equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad u(t,0) = 0, \quad \frac{\partial u}{\partial x}(t,1) - u(t,1) = 0.$$

Compute all nontrivial solutions of the form $u(t,x) = e^{\lambda t}v(x)$; consider the cases $\lambda > 0$, $\lambda = 0$, and $\lambda < 0$ separately.

Problem 3 (20 points)

Recall the following function:

$$G_0(x,y;\xi,\eta) = -\frac{1}{2\pi}\log\|(x,y) - (\xi,\eta)\|,$$

where $\|\cdot\|$ denotes the Euclidean norm. Use this function and the method of images to construct the Green's function for Poisson's equation on the domain $\Omega = \{(x,y) \in \mathbb{R}^2 : y > x\}$.

Please turn over for problems 4 and 5!

Problem 4 (15 points)

Use Fourier transforms to solve the following heat equation:

$$\frac{\partial u}{\partial t} = \gamma \frac{\partial^2 u}{\partial x^2}, \quad t > 0, \quad -\infty < x < \infty,$$

where $\gamma > 0$ and the initial condition is given by $u(0,x) = \delta(x - \xi)$.

Problem 5 (15 points)

Assume that $v(t,x) = e^{\alpha\varphi(t,x)}$ satisfies the following heat equation:

$$\frac{\partial v}{\partial t} = \gamma \frac{\partial^2 v}{\partial x^2},$$

where $\gamma > 0$. Show that for $\alpha = -1/2\gamma$ the function $u = \varphi_x$ satisfies Burgers' equation:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \gamma \frac{\partial^2 u}{\partial x^2}.$$

End of test (90 points)

Solution of problem 1 (8 + 6 + 6 = 20 points)

(a) Assume $t \mapsto x(t)$ satisfies the equation

$$\frac{dx}{dt} = u(t, x(t)),$$

and define the function

$$h(t) = u(t, x(t)).$$

Differentiating h with respect to t and using the chain rule gives

$$\frac{dh}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt} = \frac{\partial u}{\partial t} + u(t, x(t))\frac{\partial u}{\partial x} = 0.$$

(6 points)

We conclude that u is constant along a characteristic curve.
(2 points)

(b) Let $x(t)$ be a characteristic curve through $(0, y)$. Since the solution u is constant along the curve, we have

$$\frac{dx}{dt} = u(t, x(t)) = u(0, x(0)) = u(0, y) = f(y).$$

(3 points)

Therefore, the characteristic curve is a straight line given by the equation

$$x(t) = f(y)t + y.$$

(3 points)

(c) Assume that the characteristic line through the point $(0, y)$ also passes through the point $(\bar{t}, \bar{x})$. Then $\bar{x} = y\bar{t} + y$ and thus $y = \bar{x}/(\bar{t} + 1)$.
(3 points)

Since u is constant along a characteristic curve, we obtain

$$u(\bar{t}, \bar{x}) = u(0, y) = f(y) = y = \frac{\bar{x}}{\bar{t} + 1}.$$

Dropping the bars gives the explicit solution formula for u.
(3 points)

Solution of problem 2 (20 points)

The ansatz $u(t,x) = e^{\lambda t} v(x)$ gives the following boundary value problem for v:

$$v''(x) - \lambda v(x) = 0, \quad v(0) = 0, \quad v'(1) - v(1) = 0.$$

(2 points)

The case $\lambda = -\omega^2 < 0$ gives

$$v(x) = a\cos(\omega x) + b\sin(\omega x).$$

The boundary condition at $x = 0$ implies that $a = 0$. The boundary condition at $x = 1$ implies that

$$b(\omega\cos(\omega) - \sin(\omega)) = 0.$$

The equation $\tan(\omega) = \omega$ has countably many solutions $\omega_k > 0$. These give the nontrivial solutions

$$v_k(x) = \sin(\omega_k x), \quad k = 1, 2, 3, \ldots$$

(8 points)

The case $\lambda = 0$ gives $v(x) = a + bx$. The boundary conditions imply that $a = 0$ and that b is arbitrary. Setting $b = 1$ gives the nontrivial solution $v_0(x) = x$.
(3 points)

The case $\lambda = \omega^2 > 0$ gives

$$v(x) = a\cosh(\omega x) + b\sinh(\omega x)$$

The boundary condition at $x = 0$ implies that $a = 0$. The boundary condition at $x = 1$ implies that

$$b(\omega\cosh(\omega) - \sinh(\omega)) = 0.$$

For a nontrivial solution we need $\tanh(\omega) = \omega$ which has no positive solutions. Therefore, we do not get nontrivial solutions in this case.
(7 points)

Solution of problem 3 (20 points)

The Green's function is constructed by setting

$$G(x,y;\xi,\eta) = G_0(x,y;\xi,\eta) + z(x,y;\xi,\eta),$$

where z is harmonic on Ω and satisfies $z = -G_0$ on $\partial\Omega$.
(5 points)

To a point $(\xi,\eta) \in \Omega$ we associate an image point $(\xi',\eta') \in \mathbb{R}^2 \setminus \overline{\Omega}$ and set

$$z(x,y;\xi,\eta) = \frac{a}{2\pi} \log \|(x,y) - (\xi',\eta')\| + \frac{b}{2\pi}.$$

This choice guarantees that z is harmonic in Ω. Now we have to determine the constants a and b such that the condition $z = -G_0$ on $\partial\Omega$ is satisfied.
(5 points)

The boundary of Ω is given by $\{(x,x) : x \in \mathbb{R}$. If we define $(\xi',\eta') = (\eta,\xi)$, which is the reflection of (ξ,η) through the line $y = x$, then we have

$$\|(x,x) - (\xi',\eta')\| = \|(x,x) - (\eta,\xi)\| = \|(x,x) - (\xi,\eta)\|$$

for all $x \in \mathbb{R}$.
(5 points)

This implies that the condition $z = -G_0$ on $\partial\Omega$ is satisfied when $a = 1$ and $b = 0$. Therefore, the Green's function is given by

$$\begin{aligned} G(x,y;\xi,\eta) &= -\frac{1}{2\pi} \log \|(x,y) - (\xi,\eta)\| + \frac{1}{2\pi} \log \|(x,y) - (\eta,\xi)\| \\ &= \frac{1}{4\pi} \log \frac{(x-\eta)^2 + (y-\xi)^2}{(x-\xi)^2 + (y-\eta)^2}. \end{aligned}$$

(5 points)

Solution of problem 4 (15 points)

Taking Fourier transforms gives the ordinary differential equation

$$\frac{d\widehat{u}}{dt} = -\gamma k^2 \widehat{u}.$$

(3 points)

The solution is given by

$$\widehat{u}(t,k) = \widehat{u}(0,k)e^{-\gamma k^2 t} = \frac{e^{ik\xi}}{\sqrt{2\pi}} e^{-\gamma k^2 t}.$$

(3 points)

Using the table of Fourier transforms with $a = 1/4\gamma t$ gives

$$\mathscr{F}\left[e^{-ax^2}\right] = \frac{e^{-k^2/(4a)}}{\sqrt{2a}},$$
$$\mathscr{F}\left[e^{-x^2/4\gamma t}\right] = \sqrt{2\gamma t}\, e^{-k^2 t},$$
$$\mathscr{F}^{-1}\left[e^{-k^2 t}\right] = \frac{1}{\sqrt{2\gamma t}} e^{-x^2/4\gamma t}.$$

(6 points)

Finally, by using the translation property of the Fourier transform, we obtain

$$u(t,x) = \frac{1}{\sqrt{2\pi}} \mathscr{F}^{-1}\left[e^{-ik\xi} e^{-k^2 t}\right] = \frac{1}{2\sqrt{\pi\gamma t}} e^{-(x-\xi)^2/4t}.$$

(3 points)

Solution of problem 5 (15 points)

If $v_t = \gamma v_{xx}$, then φ must satisfy the equation

$$\varphi_t = \gamma\varphi_{xx} + \alpha\gamma\varphi_x^2.$$

(5 points)

Differentiating with respect to x gives

$$\varphi_{tx} = \gamma\varphi_{xxx} + 2\alpha\gamma\varphi_x\varphi_{xx}.$$

(5 points)

Setting $u = \varphi_x$ and $\alpha = -1/2\gamma$ gives

$$u_t + uu_x = \gamma u_{xx}.$$

(5 points)